On Chern-Weil theory of groups preserving a contact form

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Contact manifolds

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Example: the unit sphere $S^{2n+1} \subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$
– the standard contact structure $\xi_{\text{std}} = TS^{2n+1} \cap i(TS^{2n+1})$ ("complex tangencies")
– if $(x_1, y_1, \dots, x_{n+1}, y_{n+1})$ are the euclidean coordinates on $\mathbb{R}^{2n+1}$, then a contact form is

$$\alpha_{\text{std}} = \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j)|_{TS^{2n+1}}.$$

Contact transformation groups

Let $(M, \xi = \ker \alpha)$ be a contact manifold.

- the group of contactomorphisms

$$\begin{aligned}\mathrm{Cont}(M,\xi) &= \{f \in \mathrm{Diff}(M) \mid Tf(\xi) = \xi\} = \\ &= \{f \in \mathrm{Diff}(M) \mid f^*\alpha = \lambda \cdot \alpha \text{ for some} \\ &\qquad\qquad \lambda \in C^\infty(M) \text{ everywhere nonzero}\},\end{aligned}$$

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I study how $(M, \xi = \ker \alpha)$ can fibre over other manifolds in a (strictly) contact way.

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- The contact frame bundle of (1) is a fibre bundle $\mathrm{CFr}E \to B$ with fibre $\mathrm{CFr}E_b = \mathrm{Diff}((M, \alpha), (M_b, \alpha))$, where $b \in B$.

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$\mathrm{Cont}(M, \alpha)$ acts on $\mathrm{CFr} E$ from the right by composition of mappings and this action is free and transitive on fibres, so $\mathrm{CFr} E \to B$ is a **principal Cont**(M, α)**-bundle ...**

Principal bundles and characteristic classes

- Let G be a Lie group. A principal G-bundle is a fibre bundle $P \to B$ together with a right action of G on P which is free and transitive on fibres. In particular, the typical fibre is diffeomorphic to G itself.

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Goal: describe $H^*(\mathrm{BCont}(M, \alpha))$ – characteristic classes for principal $\mathrm{Cont}(M, \alpha)$-bundles / strictly contact fibre bundles with typical fibre (M, α)

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Then

$$R_{d\theta}\colon \Lambda^2 TP \times \ldots \times \Lambda^2 TP \xrightarrow{d\theta \times \ldots \times d\theta} \mathfrak{g} \times \ldots \times \mathfrak{g} \xrightarrow{R} \mathbb{R}$$

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is a $2k$-form on P, which descends to a **closed** $2k$-form $\widetilde{R}_{d\theta}$ on B. Its de Rham cohomology class $\chi_R(P) := [\widetilde{R}_{d\theta}] \in H^{2k}_{\mathrm{deR}}(B; \mathbb{R})$ is **independent of** the choice of the connection θ and so the assignment "$(P \to B) \mapsto \chi_R(P)$" defines a characteristic class.

Two examples

Example

1. Let $G = \mathrm{GL}(n, \mathbb{R})$, $\mathfrak{g} = \mathsf{Mat}(n; \mathbb{R})$. Consider the invariant polynomials $R_k(A_1, \ldots A_k) = \mathrm{tr}(A_1 \cdot \ldots \cdot A_k)$, $k \in \mathbb{N}$.

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2. Let $G = \mathrm{Cont}(M, \alpha)$. Its Lie algebra is the algebra of vector fields $X\colon M \to TM$ which satisfy $\mathcal{L}_X \alpha = 0$. Applying α, we get a function $\alpha(X)\colon M \to \mathbb{R}$.

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   For each $k \in \mathbb{N}$ we then have an invariant polynomial

$$R_k(X_1, \dots, X_k) = \int_M \alpha(X_1) \cdot \ldots \cdot \alpha(X_k)\, \alpha \wedge (d\alpha)^n$$

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But $\mathrm{Cont}(M, \alpha)$ is an **infinite-dimensional** Lie group and so one has to be more careful with the theory, e.g. does there exist a principal connection form?! Yes, it does, even a "nice" one.

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Theorem (OS): Let $(M, \xi = \ker \alpha)$ be a closed connected **regular** contact manifold and let $\varphi \colon S^1 \to \mathrm{Cont}(M, \alpha)$ be the S^1-action of the Reeb flow of α. Then the induced map on the cohomology of the classifying spaces

$$(\mathrm{B}\varphi)^* \colon H^*(\mathrm{BCont}(M, \alpha); \mathbb{R}) \to H^*(\mathrm{B}S^1; \mathbb{R}) \cong \mathbb{R}[t]$$

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Idea of proof: Start with a principal S^1-bundle $P \to B$ and construct the associated bundle $E = P \times_{S^1} M \to B$ with fibre (M, α). This will be a strictly contact bundle. Pull back the contact classes $\chi_k(E)$ to $P \to B$ and show that the pull-backs are the powers $e(P)^k$ of the Euler class of P.